

Chapter 1

Complex networks analysis in socioeconomic models

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Abstract This chapter aims at reviewing complex networks models and methods that were either developed for or applied to socioeconomic issues, and pertinent to the theme of New Economic Geography. After an introduction to the foundations of the field of complex networks, the present summary adds insights on the statistical mechanical approach, and on the most relevant computational aspects for the treatment of these systems. As the most frequently used model for interacting agent-based systems, a brief treatment of the statistical mechanics of the classical Ising model on regular lattices, together with recent treatments of the same model on small-world Watts-Strogatz and scale-free Albert-Barabási complex networks is included. Other sections of the chapter are devoted to applications of complex networks to economics, finance, spreading of innovations, and regional trade and developments. The chapter also reviews results involving applications of complex networks to other relevant socioeconomic issues, including results for opinion and

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citation networks. Finally, some avenues for future research are introduced before summarizing the main conclusions of the chapter.

1.1 Introduction

The foundation of the field of network topology dates back to the 18th century with the seminal work in graph theory of Euler [1] devoted to the celebrated problem of Königsberg bridges, and includes several important contributions in the last two centuries like Cayley trees [2] and the theory of random graphs due to Erdős and Rényi [3]. However, it was not until the late nineties that complex networks with specific structural features valid for the description of short path lengths, highly clustered [4], and even heterogeneous networks [5] were introduced, which opened what could be called the contemporary era of network theory. The amount of papers published since then has never ceased to increase exponentially as network theory started to be applied to fields like physics, biology, computer science, sociology, epidemiology, and economics among others. It is now recognized that a network is always the skeleton of any complex system, so it is by no means an exaggeration to say that network theory has become one of the cornerstones of the theory of complex systems. All this activity has been excellently reviewed in several occasions during the last decade (see for e.g. Refs. [6–10]), and it is now widely accepted that the dynamics of many complex systems corresponds to emergent phenomena associated to the large scale fluctuations of some real network.

The effects of the topological properties of networks on dynamical processes is also a matter of intense research inside the field of complex networks. Some of these dynamical processes are the evolution of the network itself, spreading processes in agent-based systems (epidemics in a population, rumor spreading), opinion formation, cultural assimilation, voting processes, or decision making on competing for limited resources (for a review, see Boccaletti et al. [9]). Specifically, the description of the evolution of the network is very important on itself, as real networks evolve in time. For that, one has to follow the evolution of networks, through the number of vertices, the number, weight and direction of links, and through other characteristic quantities mentioned below, as seen in [9, 11–13].

In the field of economics and economic geography, a complete understanding of economic dynamics requires the understanding of its agent-based underlying structure and the interactions that give rise to the observed emergent spatial and temporal organizations, which are definitively more than the sum of its individual components (for further details see e.g. Ausloos and coworkers in this book [14]). The view of the economy as an evolving complex agent-based system, hereafter identified with a network, is currently gaining consensus among the scientific community (see e.g. Refs. [15–17]). On the basis of this network-based structural view, the application of the methods of physics of complex systems (statistical physics, nonlinear physics, and so forth) has allowed to gain new insight on the economic realm, even leading to the foundation of a new branch of Physics itself, the so called Econophysics.

The purpose of this chapter is to summarize some applications that complex network theory has found in economic and social studies. Before reviewing some applications of complex networks to economic issues (financial markets, spreading of innovations, economic geography, regional trade and development....), the main results that have been developed up to now in the field of statistical mechanics of complex networks and their computational analysis will be briefly summarized. The chapter is closed with complex-network-based descriptions of other social networks, a brief description of future trends in the field of socio-economic applications of complex networks and our conclusions.

1.2 Summary of statistical mechanics of complex networks

As mentioned previously, in an attempt to reproduce the success in describing regular systems (solid state physics, phase transitions and so forth), statistical mechanics has been applied to heterogeneous systems through the formalism of complex networks, representing the constituents by means of vertices and their interactions by a set of edges. The description of these objects involves their topology and dynamic evolution, as well as different dynamic processes that take place over them. One consideration consists in tying the structure of the network and its intrinsic dynamics [7]. Another concerns the structural changes in the network due to the dynamical processes themselves. As we have previously mentioned, several excellent reviews have been published during the past decade on the structure and dynamics of complex networks, as well as on dynamic processes that take place in these topological objects. However, for the interest of the reader, in this section the main topics of the field will be briefly summarized.

From a formal point of view a network (graph) is a pair (V, E) , where V is a set of nodes (vertices), and E is a set of links (edges), which are identified by two nodes that represent the source and the end of the link (edge). The most used method for the representation of networks with N nodes is the *adjacency matrix* $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, whose rows and columns represent the nodes of the network, and whose terms $a_{ij} > 0$ represent the weight of the link from node i to node j . The absence of links is given by zero elements $a_{ij} = 0$. Therefore, properties of the network are reflected in the properties of the adjacency matrix. Network links are called directed if matrix A is not symmetric, which means that there exists at least one pair of indices i, j such that $a_{ij} \neq a_{ji}$. This includes as particular case graphs with upper triangular adjacency matrices, i.e. such that $a_{ij} > 0$ and $a_{ji} = 0$. In graph theory, such networks are named *directed acyclic graphs*. Undirected networks are represented by symmetric matrices ($A = A^T$). A source i is a node having only outgoing links (i.e. $\forall k \in V$ there are no pairs $(k, i) \in E$). By contrast, a sink j is a node having only incoming links (i.e. $\forall k \in V$ there are no pairs $(j, k) \in E$). A path from node i_{h_1} to node i_{h_k} in the network is a sequence of nodes $i_{h_1}, i_{h_2}, \dots, i_{h_k}$ such that $a_{i_{h_j}, i_{h_{j+1}}} \neq 0$, and can be detected through the power of the adjacency matrix A^{h_k} .

Paths are relevant for studying diffusion processes as well as the relevance of nodes. A network is called (*strongly*) *connected* if a path exists between any pair of nodes, considering the direction of the links, and it is termed (*weakly*) *connected* if a path exists among any pair of nodes when ignoring the direction of the links, which requires that the adjacency matrix be symmetric. In some applications the weight assigned to each link is 1, since the presence of the edge is relevant, but not its specific weight. In this case the matrix A is symmetrized setting $a_{ji} = 1$ for all nodes i, j such that $a_{ij} = 1$. In applications of networks to problems in which the weight of the link is relevant, whether symmetrization is applied depends on the objectives.

The main difference between network theory and graph theory lies in their targets. Naturally, algorithms first developed in the field of graph theory are useful and currently used also for complex networks and in other fields related to dynamical systems and the discretization of maps, like symbolic images [18, 19].

1.2.1 Main measures for complex networks

Networks can be classified according to a relatively large number of criteria. Taking into account the distribution of the number of links per node, networks can be classified as purely random networks (Poisson distributed), exponentially distributed small-world networks, and scale-free networks. According to the directionality of contacts they can be classified directed or undirected graphs. According to the heterogeneity in the capacity and the intensity of the connections, networks can be classified into weighted or unweighted networks depending on whether different weights are associated to their edges. Sparse and fully connected networks differ in the fraction of interconnected nodes. Finally, depending on their time evolution, networks can be classified into static and evolving.

The description of the topology of complex networks is based on several concepts and parameters that measure different features of these topological structures and macroscopic characteristics of the networks (a more detailed treatment can be found in [20]). The most important of these are:

1. Average path length: no proper metric space can be defined for complex networks (usually hidden metric spaces are defined [21]), and the (chemical) distance between any two vertices l_{ij} is defined to be the number of steps from one point to the other following the shortest path. In many real networks, the average distance between two nodes, i.e. the average path-length, $\langle l \rangle$, is relatively small as compared to the total number of nodes in the network. In fact, in a regular lattice, a topological structure which can be generated from a basis for the vector space by forming all linear combinations with integer coefficient and where all the nodes are connected to the same number of neighbors, the average path length scales with the number of nodes in the network, N , as $\langle l \rangle \sim \sqrt{N}$, while in a small-world network with long-distance shortcuts, $\langle l \rangle \sim \log N$, so the separation between any two nodes is usually very small. This property is behind the

small-world concept first studied by Milgram in his 1967 seminal paper [22], and it is by no means included in conventional regular lattices. The connectivity can also be measured by means of the diameter of the graph, d , defined as the maximum distance between any pair of its nodes. Dijkstra algorithm [23] is the most often used for this calculation.

2. Degree distribution, $p(k)$, measuring the probability that a given node has k connections to other nodes. This is probably the most important property of networks, and it is behind their classification as exponentially distributed Watts-Strogatz (WS) networks ($p(k) \sim e^{-\alpha k}$) and scale-free Barabási-Albert (BA) networks ($p(k) \sim k^{-\gamma}$). For purely random Erdős-Renyi networks, $p(k)$ is a binomial distribution (so also belonging to the exponentially distributed class,

$$p(k) = \binom{N}{k} p^k (1-p)^{N-k} \quad (1.1)$$

Sparse networks are those for which the average degree remains finite when $N \rightarrow \infty$, and for real networks, $\langle k \rangle \ll N$.

3. Clustering: The clustering coefficient c_i of a vertex i is given by the ratio between the number e_i of triads -connected subsets of three network nodes- sharing that vertex, and the maximum number of triangles that the vertex could have. Alternatively, this coefficient is the ratio between the number E_i of edges that actually exist between the k_i neighbors of vertex i and the maximum number $k_i(k_i - 1)/2$. The clustering coefficient provides a measure of the local connectivity structure of the network. This coefficient usually takes large values in social networks, contrary to what happens in random graphs [6]. The average cluster coefficient is given by $\langle c \rangle = \sum_i c_i / N$ and the clustering spectrum by $\langle c(k) \rangle = \sum_i \delta_{k, k_i} c_i / N p(k)$, where N is the number of vertices in the network, and $p(k)$ is the degree distribution defined below. A graph is considered to be small-world if its clustering coefficient is considerably greater than that of a random graph built on the same node set and the average path length is approximately the same as that of the corresponding random graph. One may point out that the clustering coefficient of triangular Erdős-Renyi networks, i.e. uncorrelated random graphs, is very high by construction, but, contrary to intuition, it is different from 1 in general.
4. The overlapping index [24] measures the common number of neighbors of the i and j nodes, i.e. how many triads have a common basis. Moreover, a network transitivity is the probability that two neighbors of a node have themselves a link between them. In topological terms, it is a measure of the density of triads in a network.

1.2.2 Additional parameters and concepts

In addition to the ones mentioned above, there are a lot of other parameters and concepts that measure important properties of the topology of complex networks. A thorough treatment of these can be found in more detailed reviews like the ones cited in this chapter or in monographs like for e.g. that in Ref. [8]. For brevity, we shall mention just a few of them of particular interest.

1. Centrality of a node [25]. Many measures are gathered under the label “Centrality”. The node degree, i.e. its number of contacts to other nodes of the network, is one of them. Another one is the so called betweenness of a vertex b_i or an edge b_{ij} , which is the number of shortest paths that pass through the vertex i (edge (i, j)), for all the possible pairs of vertices in the network.
2. Correlations in networks: The correlations usually found in real networks (i.e. the fact that the degrees of the nodes at the ends of a given vertex are not in general independent) are measured by means of the distribution $p(k | k') = k' p(k') / \langle k \rangle$, representing the conditional probability that an edge that has one node with degree k has a node with degree k' at the other end. In a correlated network, this distribution depends both in k and k' , while in an uncorrelated network, it depends only on k' . An alternative measure of correlations is given by the average degree of the nearest neighbors of the vertices of degree k [26],

$$\langle k_{nn} \rangle = \sum_k k' p(k | k') \quad (1.2)$$

The network is said to be correlated if this parameter depends on k . Other measures are the closeness centrality or the flow-betweenness centrality. When $\langle k_{nn} \rangle$ increases with k the network is called assortative, while if $\langle k_{nn} \rangle$ is a decreasing function of k , the network is called disassortative. Then, the assortativity coefficient [27] measures a network property through the Pearson correlation coefficient of the degrees at either ends of an edge. Finally, one must recall that $p(k | k')$ and $p(k)$ are not independent, but, due to the conservation of edges, they are related by a degree detailed balance condition [28]

$$k p(k' | k) = k' p(k') p(k | k') \quad (1.3)$$

3. k -shells. A k -shell of a graph G is a connected subgraph of G in which all vertices have degree at least k . Equivalently, it is one of the connected components of the subgraph of G formed by repeatedly deleting all vertices of degree less than k . The k -core of a graph G is the k -shell with the maximum k . The concept of k -shells and k -cores was introduced to study the clustering structure of social networks and to describe the evolution of random graphs; it has also been applied in bioinformatics and network visualization. In Economics and Finance it has been applied to the corporate ownership network, to the international trade network, and to the network of shareholders [29, 30, 104].

4. Nestedness index [31]. It indicates the likelihood that a node is linked to the neighbors of the nodes with larger degrees. The mean topological overlap between nodes [32] has been introduced to quantify nestedness.

1.2.3 Ising model on complex networks

Several classical problems of statistical mechanics have been now studied using complex networks. In particular, the mean-field solution for the average path length and for the distribution of path lengths in small-world networks have been reported by Newman et al. [33]. On the other hand, the mean-field solution of the Ising model [34] on a small-world complex network has been contributed by several authors [35, 36, 38, 39], and by Bianconi on a BA network [40] in the first half of the last decade. Viana-Lopes [38] solved the 1D Ising model on a small-world network, and Herrero [39] considered the ferromagnetic transition for the Ising model in small-world networks generated by rewiring 2D and 3D lattices. Due to its interest for socioeconomic researchers, some of the main results reported in these contributions are recalled below.

The Ising model [34], originally introduced for the study of ferromagnetism, is the simplest paradigm of order-disorder transitions. Undoubtedly it represents one of the major milestones in the development of statistical mechanics of interacting systems and phase transitions, and it is at the basis of a plethora of applications and generalizations reported throughout the 20th century. This is a discrete model in which spins (agents) with two possible states (± 1) are placed in the nodes of a graph (normally a lattice) and are allowed to interact with their nearest-neighbours, being probably the simplest model to exhibit a phase transition. The one dimensional problem was solved by Ising himself in 1925 [34], and the exact solution to the 2D problem was reported by Onsager in 1944 [41]. The model Hamiltonian for the N spins $s_i = \pm 1$ on the nodes of the graph is given by

$$H = - \sum_{i,j} J_{i,j} s_i s_j - \sum_i h_i s_i \quad (1.4)$$

h_i being the local external (magnetic) field, and J_{ij} being nonzero only for those pairs of spins connected by a link. If $J_{ij} > 0$ then parallel orientations of spins are energetically favored (ferromagnetism), while for $J_{ij} < 0$ antiparallel orientations are preferred (antiferromagnetic case). In social physics applications these cases correspond, respectively, to consensus/dissensus-oriented models. For the nearest neighbours 1D problem, eq. (1.4) can be rewritten as

$$H = -J \sum_{i=1}^N s_i s_{i+1} - \sum_{i=1}^N h_i s_i, \quad (1.5)$$

with $s_{N+1} = s_1$ as usual for periodic boundary conditions. The canonical partition function -the normalization factor of the probability density in the phase space of

the system's microstates- is straightforwardly calculated from the above equation as:

$$\begin{aligned}
 Z(\beta) &= e^{-\beta H(\sigma)} \\
 &= \sum_{s_1 \dots s_N} e^{\beta h s_i} e^{\beta s_i s_{i+1}} \\
 &= \sum_{s_1 \dots s_N} \prod_{i=1}^N \Delta_{s_i s_{i+1}}
 \end{aligned} \tag{1.6}$$

where $\beta^{-1} = k_B T$ represents the thermal energy and $\Delta_{s_i s_{i+1}} = \exp(\beta h s_i) \exp(\beta s_i s_{i+1})$ $\exp(\beta h s_{i+1})$ is the transfer matrix. Thus, using conventional matrix algebra, we can obtain the partition function in terms of the eigenvalues of the matrix Δ^N , $\lambda_i (i = 1, 2)$, as:

$$\begin{aligned}
 Z(\beta) &= \text{Tr}(\Delta^N) \\
 &= \lambda_1^N \left[1 + \left(\frac{\lambda_2}{\lambda_1} \right)^N \right]
 \end{aligned} \tag{1.7}$$

and the associated Helmholtz free energy per spin (a thermodynamic potential comprising all the relevant thermodynamic information about the system and governing equilibrium and stability at constant temperature and volume), $f(\beta)$, as:

$$\begin{aligned}
 -k_B T f(\beta) &= \lim_{N \rightarrow \infty} \frac{Z(\beta)}{N} \\
 &= \ln \left(e^{\beta J} \cosh(\beta h) + \sqrt{e^{2\beta J} \sinh^2(\beta h) + e^{-2\beta J}} \right)
 \end{aligned} \tag{1.8}$$

Moreover, one can prove (see for example Ref. [42] for an elegant treatment of the topic) that spin-spin correlations are given in this model by:

$$\langle s_i s_j \rangle = e^{-r_{ij}/\xi} \tag{1.9}$$

where $\xi = a/|\ln \tanh(\beta J)|$, with a being the lattice spacing between spins. It is well-known [42] that a 1D system exhibits no phase transition in the absence of long-range interactions. In 2D systems this ferromagnetic transition ($J > 0$) is registered, and it was Onsager [41], who obtained the partition function for the vanishing external magnetic field case, and Yang [43] who calculated the magnetization in the ferromagnetic phase given by

$$M = \left\{ 1 - \left[\sinh \left(\log(1 + \sqrt{2}) \frac{T_c}{T} \right) \right]^{-4} \right\}^{1/8} \tag{1.10}$$

Here $k_B T_c = 2J / \log(1 + \sqrt{2}) \simeq 2.27J$ is the critical temperature where the ferromagnetic (consensus) transition takes place.

Contrarily to their physical homologues, where interactions are usually of limited range, in social systems long-range connections between agents are very frequently registered. In order to preserve an Ising-based descriptions of these systems, it is necessary to generalize the formalism so as to include the existence of long-range correlations between agents, and that is most conveniently done by means of complex networks.

The solutions of the 1D Ising model on a small-world network has been reported by Viana-Lopes et al. [38], and on a scale-free network by Bianconi [40], the latter in the mean field approximation. In the former work, the authors exactly solved a one-dimensional Ising chain with nearest neighbor interactions and random long-range interactions, obtaining a phase transition of a mean-field type. The authors considered a 1D lattice in which the bonds are rewired at random with a probability p in a Watts-Strogatz fashion, and they used a Hamiltonian where they allowed the existence of different short-range (chain, J) and long-range (I) interactions,

$$H = -J \sum_{i=0}^N s_i s_j - I \sum_{ij \in S} s_i s_j - h \sum_{i=1}^{N-1} s_i \quad (1.11)$$

where the set S includes the $N_b = Np$ shortcut pairs of nodes connected by a long-range connection. Even for small values of the shortcut probability p , a dramatic increase in the connectivity of the network is registered, which has a deep influence in the thermodynamics of the Ising problem. Particularly, Viana-Lopes et al. [38] proved that a phase transition exist in this 1D problem with a transition temperature given by

$$t_J^d (1 + 2t_I) = 0 \quad (1.12)$$

where $d = 1/2p$, $t_J = \tanh(\beta J)$ and $t_I = \tanh(\beta I)$. The authors analyzed the behavior of this temperature in the limits of shortcut bonds stronger than chain bonds ($pI \rightarrow \infty$ for any finite p) and of chain bonds much stronger than shortcut bonds ($pI \rightarrow 0$) and obtained:

$$\begin{aligned} T_c &= \frac{2J}{\ln(1/p \ln 3)} & pI \rightarrow \infty \\ T_c &= \frac{2J}{\ln\{J/[pI \ln(J/pI)]\}} & pI \rightarrow 0 \end{aligned} \quad (1.13)$$

On the other hand, the Ising model on a BA network has been solved in the mean-field approximation by Bianconi in Ref. [40], as previously mentioned. The author showed that the mean-field solution of the Ising model in this type of network can be treated as a Mattis model, a simple solvable model of the spin glass in which Ising spins interact via unfrustrated random exchange interactions [44]. The author considered a BA network of N spins constructed iteratively with the constant addition of new nodes with m connections and a Hamiltonian

$$H = -J \sum_{i,j} \varepsilon_{ij} s_i s_j - \sum_{i=1}^N h_i s_i, \quad (1.14)$$

Here $\varepsilon_{ij} = \langle A_{ij} \rangle = k_i k_j / 2mN$ is the average of the adjacency matrix over many copies of the network [40]. The mean-field solution of the Hamiltonian for the order parameter S is

$$\begin{aligned} S &= \frac{1}{2mN} \sum_{i=1}^N k_i \langle s_i \rangle \\ &= \frac{1}{2mN} \sum_{i=1}^N k_i \tanh [\beta (J k_i S + h_i)] \end{aligned} \quad (1.15)$$

where one can see that the effective mean-field acting on a spin is determined not only by the external field and the interaction strength, as in conventional Ising model, but also by the connectivity of the network nodes. The above equation resembles that of the Mattis model with the substitution of the quenched random variables ξ_i in Ref. [44] by the node degree k_i . Bianconi was able to prove from the above that the effective critical temperature is given by

$$T_c = \frac{mJ}{2} \ln(N), \quad (1.16)$$

i.e. it increases linearly with the interaction J and logarithmically with the number of nodes in the system, in agreement with previously reported numerical simulations (see Ref. [40] for further details).

Finally, it is worth mentioning that the existence of phase transitions in 1-D systems with long-range interactions has been proved by means of Monte Carlo simulations by Pekalski [37], who demonstrated that even a small fraction of long-distance shortcuts induces ordering of the system at finite temperatures and provided the dependence of the magnetization and the critical temperature on the concentration of the small world links.

1.2.4 A special case: Bipartite networks reductions

For our present purposes, it must be here emphasized that when discussing interacting economic entities, it is paramount to discriminate N -body correlations that are intrinsic N -body interactions from those that merely develop from lower-order interactions, like the 2-body interactions of the Ising model. This issue is directly related to a well-known problem in complex network theory, i.e. the projection of bipartite networks composed of two kinds of nodes, onto unipartite networks, i.e., composed of one kind of node. This property of bipartiteness is a special case of disassortativity. A network is called bipartite if its vertices can be separated into two

sets such that edges exist only between vertices of different sets. [45]. Bipartite networks are well known in graph theory and operations research, where the delivery problem from N sources to M sinks is well studied, and finds a first application in Economics to the problem of finding the optimal supply of goods (in the N sources) to accomplish the demand function of the M consumers (the M sinks) [46]. The model may vary to consider the optimal location and geographical distance among economic activities (the N sources) and the customers (the M sinks). Further recent applications to financial networks relate the set of N companies to the set of their M directors, who are persons, so the two sets are naturally describing two very different categories. This is naturally a bipartite graph, and there is a link among a company and a person if he/she is in the administrative board of the company. This network can be represented through a matrix $A \in \mathbb{R}^{N \times M}$ that is the starting point for the study of ties among companies given by the presence of the same directors in their boards (AA^T), or the connections among persons due to belonging to the same boards ($A^T A$) [47–51].

This formalism and a coarse graining description of bipartite networks from a statistical mechanics approach can be found in [45, 52–54]. Formally, the bipartite structure of e.g., quantities or prices vs. producers may be mapped exactly on the *vector* of matrices \mathcal{M} defined by:

$$\mathcal{M} = [M_{a_1}^1, M_{a_1 a_2}^2, M_{a_1 a_2 a_3}^3, \dots, M_{a_1 \dots a_{n_p}}^{n_p}] \quad (1.17)$$

where \mathbf{M}^j is a square n_p^j matrix that accounts for all quantities (at some price) j produced by producer P . For example, $M_{a_1}^1$ and $M_{a_1 a_2}^2$ represent respectively the total number of goods produced by a_1 alone, and the total number of goods produced by the pair (a_1, a_2) .

It is important to point out that the vector of matrices \mathcal{M} describes the bipartite network without approximation, and that it reminds of the Liouville distribution in phase space of a Hamiltonian system. Accordingly, a relevant macroscopic description of the system relies on a coarse-grained reduction of its internal variables. The simplest reduced matrix is the single producer matrix:

$$R_{a_1}^1 = M_{a_1}^1 + \sum_{a_2} M_{a_1 a_2}^2 + \sum_{a_2} \sum_{a_3 < a_2} M_{a_1 a_2 a_3}^3 + \dots + \sum_{a_2} \dots \sum_{a_j < a_{j-1}} M_{a_1 \dots a_j}^j + \dots \quad (1.18)$$

that is a vector whose elements $R_{a_j}^1$ denote the total number of goods produced by a_j . The second order matrix:

$$R_{a_1 a_2}^2 = M_{a_1 a_2}^2 + \sum_{a_3} M_{a_1 \dots a_3}^3 + \dots + \sum_{a_3} \dots \sum_{a_j < a_{j-1}} M_{a_1 \dots a_j}^j + \dots \quad (1.19)$$

Its elements represent the total number of quantities produced by the pair (a_1, a_2) .

Remarkably, this matrix reproduces the usual projection method and obviously simplifies the bipartite structure by hiding the effect of higher order interactions. One may next discriminate between different types of triangles and discuss, e.g. the interplay between producers at the node degree level. Economic directed and

weighted networks such as payment networks (see Bougheas and Kirman in this volume) needs to be further explored.

1.3 Computational description of complex networks

The representation of graphs and networks as data structures in computer memory is by now well established, with several comprehensive and efficient implementations openly available [55]. The representation determines both the storage requirements and the efficiency of common operations on networks, and must be chosen accordingly. It must be noted that the available formats differ mostly in how edges are stored, as the most appropriate structure for storing data about vertices is relatively independent of the topology.

A particularly simple data structure for storing graphs is the adjacency matrix $A = (a_{ij})$ with as many rows and columns as edges in the graph. Among adjacency matrices, the less complex are those of simple undirected graphs with no edge weights, where the diagonal elements are always zero, $a_{ij} = a_{ji}$ and each element is 1 if the corresponding edge exists and 0 otherwise. However, when edge directionality is introduced the symmetry constraint must be abandoned, and if edges have weights they can be used as matrix entries. The main virtue of adjacency matrices, beyond the straightforwardness of their implementation, is the efficiency of adding/removing edges and checking for their existence. On the other hand, they are particularly poorly suited for representing social networks, which tend to be sparse: in a naive implementation of an adjacency matrix for a graph with N vertices, most of the N^2 entries will be zero. However, these matrices can still be used as long as they are stored in one of the formats commonly employed for sparse arrays, such as a coordinate list (list of tuples (i, j, a_{ij}) containing only those for which $a_{ij} \neq 0$) or a dictionary using the (i, j) coordinates of non-zero elements as keys [56]. Alternative, specific formats exist for graphs. For instance, in an adjacency list the set of neighbors is stored along with the rest of the data for each vertex, which keeps vertex addition and removal very efficient while ensuring that storage space scales only linearly with N .

It is not often that one finds the opportunity, or even the need, to model a real-world social network in a completely detailed fashion. Although such studies might be useful to assist political decisions, most theoretical approaches are commonly focused on exploring general phenomena, for which such overfitted models would be of little help. Instead, ensembles of random networks are built that capture just the essential features of the real networks. Of interest for social models is the fact that in most generating algorithms the structure of a network is a phenomenon that emerges from its growth dynamics. For instance, maximally random networks, where each of the $N(N-1)/2$ edges has the same probability (p) of being present regardless of the degrees of the vertices it joins, can be created using the classical Gilbert [57] algorithm, in which edges are added at random to an initially disconnected graph. The results are equivalent to those of the Erdős-Renyi [3] model: networks that

have no loop with lengths much shorter than the size of the graph. These local tree-like loops give rise to the small-world property of purely random graphs, in which the average distance between nodes grows only proportionally to the logarithm of N . Another very important phenomenon observed in these simple models is the emergence of a giant connected component, comprising a finite fraction of all nodes even in the infinite-network limit, when $p > 1$.

Both of the aforementioned properties have been observed repeatedly in real-world social networks. One of the most striking examples is a recent comprehensive study of the structure of the Facebook "friend" network, with 721 million users [58], 99.91% of them in a single connected component and an average distance between nodes of just 4.71. However, the limitations of Erdős-Renyi for describing real networks are also readily encountered when looking at the local environment of each vertex. The distribution of node clustering coefficients (number of links that exist among the neighbours of a degree- k node, normalized to the maximum number $k(k-1)/2$ that could exist) or the network clustering coefficients (fraction of the total possible 3-loops, or triads, actually present in the graph) tends to zero in the infinite-network limit and underestimates observed values by two to three orders of magnitude. This is a reflection of the fact that in real-life processes such as interactions between individuals links are more likely to be formed with other nodes in the local environment as opposed to distant ones, a bias not taken into account at all in simple random models. Another trivial limitation is that the Bernoulli processes used to build these families of graphs result in Poisson-like exponential distributions, while the shape of experimental distributions is often found to be less quickly decaying, and even scale-free.

The first problem is tackled by modified construction algorithms such as the celebrated Watts-Strogatz model [4], that takes a regular network with a ring topology (high clustering coefficient, long mean distance between nodes) as its starting point and proceeds by replacing its edges with random shortcuts with a uniform probability p . The small-world limit is quickly reached even for small values of p , whereas the clustering coefficient decays more slowly as $(1-p)^3$. Hence, Watts-Strogatz networks interpolate between the desirable properties of regular graphs and the Gilbert model. However, their degree distribution is still exponential for large k .

In contrast, the Barabási-Albert model [6] for network growth dynamics achieves a scale-free degree distribution with a $\propto k^{-3}$ fat tail by using a preferential attachment mechanism: the starting point is a small and fully connected network and at each step a new node is added and connected to m of the existing ones with probabilities proportional to their degrees. The very connected nodes present in the final network result in mean distances between nodes even shorter than those in small-world networks. Other functional choices for representing the preference for attachment with well-connected nodes are possible, and as long as they are asymptotically linear they also result in scale-free degree distributions [59], with exponents down to 2. This limit can be overcome by abandoning pure preferential attachment and introducing more drastic measures, such as merging of vertices [60], into the dynamics of the network. It must be mentioned here that, since real studies are always performed on finite networks, and fat tails may only become clear at high values

of k , deciding on whether a real network is scale-free or not can be mathematically and computationally challenging; in fact, different groups working on the same data have occasionally reached opposite conclusions [61].

Still, none of these models offer the researcher the possibility to computationally build ensembles of networks with a known degree distribution, possibly taken from experiment. Fortunately, it is straightforward to generalize, for instance, the idea behind the Erdős-Renyi model to sample graphs uniformly from the ensemble formed by all those with the desired degree distribution [62]. The key feature of such models is the fact that, by construction, the degrees of the nodes at either end of the same link are not correlated. More specifically, their joint degree distribution $p(k, k')$ can be factorized as $p_e(k) p_e(k')$; here, $p_e(k)$ is the degree distribution of an end of a randomly chosen edge, related to the node degree distribution $P(k)$ by $p_e(k) = p(k)/k$. When their degree variance is finite, these uncorrelated networks are also locally tree-like, but their clustering coefficients in finite cases approximate those of real networks much better than the ones derived from the Gilbert model. Moreover, it is still possible to derive a general condition for the emergence of a giant connected component in this setting, the Molloy-Reed criterion [63, 64], $\langle k^2 \rangle > 2 \langle k \rangle$. This criterion is met by scale-free networks, whose second moment diverges in the infinite limit, which explains their extreme resilience.

Clearly, not even the most general uncorrelated networks can capture all the varieties of real-world structures. Even though the joint distribution $P(k, k')$ is seldom available, the Pearson correlation coefficient between k and k' , known in this context as assortativity (r) can be used [27] to discriminate between correlated and uncorrelated networks. Preferential-attachment algorithms such as Barabási-Albert give rise to assortative ($r > 0$) networks, while electrical grids, for instance, are known to be disassortative ($r < 0$) [61].

A computational study of a network in the time domain will typically start with a network in an initial condition that can be empty, comprehensively sampled from real-world data or generated using one of the algorithms described above and incorporating only the necessary information. It will then proceed through a set of discrete time steps until the desired convergence is achieved. At each step, both the structure of the network and any node or edge variables may change in response to external fields, internal phenomena or interactions through the network. Several points make this kind of simulation very different from models based purely on differential equations, with a longer tradition in physics and economics. First, the computational demand of agent-based models can be formidable when compared with more aggregate treatments of similar systems; thus, automation and parallelization are critical. Even when the network dynamics can be implemented synchronously, with changes only depending on data from previous steps, to avoid race conditions, scaling can be hindered by the need to exchange large amount of data during updates between steps. Moreover, judicious choice of what aggregate variables to track to characterize the evolution of the network [11, 12] is of crucial importance to separate signal from noise. In addition to a set of basic descriptors (number of nodes and links, weights and directions if applicable, and so on) and degree and clustering distributions, more problem-specific metrics such as assortativity, betweenness cen-

tralities [25], and overlap [24] and the nestedness indices may be necessary. Finally, given the very specialized nature of network visualization, storage of data should be done in standard formats (HDF5, NetCDF) and the network itself made available in well-documented formats such as GraphML and GML to retain freedom to look at the results from different tools and environments. Third, judicious choice of what aggregate variables to track is of crucial importance to separate signal from noise. Open, scalable, general-purpose visualization tools [65–67] are currently available to ease post-processing work.

1.4 The economy as a complex adaptive system: Complex-network-based market models

In the last decades, there has been an increasing amount of effort to conceive the economy as an evolving complex system (see for example Ref. [16, 68–75]), meaning that it is a dynamic network of interactions between similar, connected agents which self-organize in order to adapt to a changing environment and maintain the organization of the macrostructure. Many of these approaches employ the complex network formalism -either as a theoretical or a computational tool- for analyzing different perspectives of the economic realm: (i) economics formalism itself, (ii) finance, and (iii) social networks applied to economics. Below, some of the main contributions in the field are reviewed.

Theoretical and computational agent-based models of economic interactions are being progressively more used by the scientific community to describe the emergent phenomena and collective behavior in economic activities arising from the interaction and coordination of economic agents [15]. The complex-network formalism is essential for most of these descriptions. This kind of approach has been used in several problems such as international trade [76], finance [77, 78], globalization [79–83] and so forth). Specifically, Schweitzer et al. [83] analyzed economic networks as tools for understanding contemporary global economy. These authors considered the ability of these objects to model the complexity of the interaction patterns emerging from the incentives and information behind agents’ behavior from which metastabilities, system crashes, and emergent structures arise. A detailed characterization of these phenomena requires, according to the authors, “a combination of time-series analysis, complexity theory, and simulation with by game theory, and graph and matrix theories”.

1.4.1 Economics: Gross Domestic Product and other macroeconomic indicators

Many contributions have been reported that consider the application of complex networks to the analysis of gross domestic product either on a national basis or from

a global perspective. Fujiwara et al. [84] considered the production network formed by a million firms and millions of supplier-customer links, showing “in the empirical analysis scale-free degree distribution, disassortativity, correlation of degree to firm-size, and community structure having sectoral and regional modules”. Moreover, Lee et al. [85] considered the influence of the global economic network topology to the spreading of economic crises, showing, by means of network dynamics, that its connectivity in the global network conditions the role of the nation in the crisis propagation together with its macroeconomic indicators.

The problem of measuring the degree of globalization of the economy is also an outstanding question. One method is to search for a measure of the clustering features through the notion of economic distance. According to Miskiewicz and Ausloos, it is possible [86] to develop a distance and graph analysis for doing so. This has been performed on the GDP of G7 countries over 1950-2003. In fact, several (4) different distance functions can be used and the results compared. Moreover, the graph method takes two forms in [86], i.e. (i) a unidirectional or (ii) a bidirectional chain. In brief, the (linear) network is allowed to grow, accumulating distances, in one or two directions.

Defining the percolation transition threshold, as the distance value at which all countries are connected to the network, it has been found that the correlations between GDP yearly fluctuations [86, 87] achieve their highest value in 1990. Hence, the globalization of the world economy was seen to disappear as early as 2005 in publications, and is so confirmed nowadays by economists.

Macroeconomic indicators other than GDP correlations can be used for such globalization/ infinite cluster search. In [87], 11 of them were investigated for the 15 original EU countries, - the data taken between 1995 and 2004. Moreover, besides the Correlation Matrix Eigensystem Analysis, a Bipartite Factor Graph Analysis is of interest for confirming the existence of stable “economic” communities. It has been interestingly found that strongly correlated countries, with respect to these 11 macroeconomic indicators fluctuations, can be partitioned into clusters, mainly based on geographic grounds.

The Moving Average Minimal Length Path algorithm has allowed a decoupling of the fluctuations. Whence a Hamiltonian representation can be formulated, given by a factor graph. Practically, that means that the Hamiltonian-Liouville machinery could be used thereafter. In fact, the Hamiltonian plays the role of a cost function.

It is somehow evident that markets can not be instantaneously correlated. In [88], forward and backward correlations, and distances, between GDP fluctuations were calculated for 23 developed countries. In this study, the network links were not only weighted but were also directed due to the arrow of time. Filtering the time-delayed correlations by successively removing the least correlated links, an evolution of the 23 countries network can be visualized. In so doing, this percolation idea-based method reveals the emergence of connections, but interestingly also of leaders and followers.

It is also relevant to know what time window has to be used when averaging properties, both in micro- and macro-economic considerations. In [89], several statistical distances between countries were calculated for various moving time windows.

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Up to 4 macroeconomic indicators were investigated: GDP, GDP/capita (GDPpc), Consumption, and Investments. In using a so called optimal one, some empirical evidence has been presented to indicate economic aspects of globalization through such indicators. The hierarchical organization of countries and their relative movement inside the hierarchy have been described.

On the practical side, there are policy implications concerning the economic clusters arising in the presence of Marshallian externalities. Relationships between trade barriers, R&D incentives and growth were identified. It is recommended that they should be accounted for designing a cluster-promotion policy [88]. Not only it should be admitted that fluctuations in macroeconomic indicators result from nearby node policies, but the effect of policy changes are not immediate. Therefore, some time averaging is necessary in order to find out, how long it takes before some policy affects some country economy and its neighboring or its connected countries. As in [89], an investigation of the weighted fully connected network of the 25 countries (nodes) forming the European Union in 2005 was presented [90] to study such time parameter effects. The links were taken to be proportional to the degree of similarity between the macroeconomic fluctuations and the GDP/capita (GDPpc) annual rates of growth between 1990 and 2005, measured by the “coefficients of determination” [90]. It has been found that the effect of the time window size for averaging and finding some robust correlations was a 7-years time window; this time interval leads to coherence in the data analysis and subsequent interpretations. A calculation of the “overlap index” [24] reveals the emergence and stability of a “hierarchy” among EU countries [89, 90].

The Cluster Variation Method for weighted bipartite networks, discussed in Sect. 2, was applied in [91]. The method allows to decompose (or expand) a “Hamiltonian” through a finite number of components, serving to define variable clusters in a given (fully connected) network. In the case studied in [91], the network was built from data representing correlations between 4 macro-economic features: Gross Domestic Product (GDP), Final Consumption Expenditure (FCE), Gross Capital Formation (GCF) and Net Exports. Two interesting features were deduced from such an analysis: (i) the *minimal entropy clustering scheme* is obtained from a coupling *necessarily* including GDP and FCE; (ii) the *maximum entropy* corresponds to a cluster which does *not explicitly include* the GDP.

A new methodological framework for investigating macroeconomic time series has been introduced in [92], based on a non-linear correlation coefficient. Features related directly to the Latin American, rather than EU, countries have been described, i.e. those Latin American countries where Spanish and Portuguese languages prevail. Again, clusters, in the networks, and emergence of a “hierarchy” have been identified through the “Average Overlap Index” [24] hierarchy scheme.

A principal component analysis has further been applied in order to observe and corroborate whether a country clustering structure truly exists [92], and the results confirmed previously expected results.

1.4.2 Finance: market correlations and concentration

In this section some literature concerning the application of complex networks to financial problems will be reviewed, without any particular attention being paid to the problem of banking risk, which is treated in other chapter of this book by Bougheas and Kirman [93]. Besides using a network structure, risk is widespread modelled through correlations. However, correlations wholly describe risk distribution function only if the phenomena under observation are Gaussian, since they correspond to the second order moment of the joint probability distribution. Higher order models could be relevant in more general situations, giving rise to deviations from the Gaussian distribution. This notwithstanding, it is worth mentioning that in option pricing, although there is evidence of deviations from the Gaussian hypothesis [94], most models used by practitioners predominantly rely on it and thus give rise to self-fulfilling prophecies [95]. There is a plethora of quantitative and econometric models to analyse option pricing, while the complex-network perspective for understanding the relative distance among systems and defining clusters is much more recent. In relation to the complex-network approach it has been remarked that filtering financial data is relevant for introducing a measure of distance based on stock market indices [96, 97]. Subsequent interdisciplinary studies show how to extract relevant information through methods first proposed in graph theory, namely Maximum Spanning tree and the Planar Maximally Filtered Graph also under dynamical adjournment of data [97–99].

Giulia:
adjournment → delay?

Yet, the correlation matrix is not the only instrument available for understanding systemic risk due to the connections among companies. Both the cross-shareholding matrix and the board of directors describe ties among companies, that may cause cascade spreading of financial crises, and raise the question of who is controlling the controller, or having a relative size on markets sufficient to pass the critical threshold of market concentration, triggering the antitrust measures [49–51, 100–104]. The development of algorithms for detecting market leaders and related optimization problems is studied also using operation research methods [105], thus confirming the interdisciplinary nature of the field. Centrality measures on networks are applied for their purpose of evidencing leaders, and consider antitrust policies for the markets. Moreover, due to the globalization of trading, the default or crisis of companies in one country may spread worldwide [29, 30, 104, 106].

As recently claimed by Lux and Westerhoff, [107] “economic theory failed to envisage even the possibility of a financial crisis like the present one. A new foundation is needed that takes into account the interplay between heterogeneous agents.” This foundation can be found in the field of complex networks, as stated by Catanzaro and Buchanan in the same issue [108]. Several other papers in the same special issue of Nature Physics contribute different applications of complex networks to the field of finance [77, 109]

Pursuing this new foundation, a great number of results have been reported in the recent past concerning the application of complex networks formalism to the analyses of financial problems. Probably, these new topological objects have not been applied to any other field of economics to a larger extent. Sampling the work

in this area we will mention the works of Onnela, Saramäki, Kaski and coworkers [110–118] for an extensive application of dynamic asset trees to financial markets, as well as to clustering, communities and correlations in these systems.

Oatley et al. [119] analyzed the political economy of global financial network using a network model, and Caldarelli et al. [77] also employed this formalism and statistical mechanics for reconstructing a financial network even from partial sets of information. On the other hand, da Cruz et al. [78] applied non-equilibrium statistical physics to a system of economic agents obeying the Merton-Vasicek model for current banking regulation and forming a network of trades by means of the exchange of an “economic energy”. The authors analyzed the propagation of insolvency (i.e. the falling of an agent below a minimum capital level) in this network and were able to prove that the avalanche sizes are governed by power-law distributions whose exponents are related to the minimum capital level. Avalanches have been proved to occur also due to behavioral aspects like the blindness to small changes in the worldwide network of stock markets [120]. Finally, we will mention the work of Bonanno et al. [121] who considered correlation-based networks of financial equities.

1.4.3 Tax evasion

The network approach has also been applied to the study of tax evasion by Westerhoff et al. [122]. The authors use the standard two-dimensional Ising model treated in Section 2 to analyze the effect of the structure of the underlying network of taxpayers on the time evolution of tax evasion in the absence of measures of control. Furthermore, it is shown that “even a minimal enforcement level may help to alleviate this problem substantially”. The number of applications of the Ising model is thus augmented suggesting an enforcement mechanism to policy-makers for reducing tax evasion.

Moreover Zaklan et al. in [123] allow tax evaders to be randomly subjected to audits, assuming that if they get caught they behave honestly during certain time. Considering different combinations of parameters, they proved that using punishment as an enforcement mechanism can effectively control tax evasion.

1.4.4 Business and spreading of innovations

Beyond the numerous applications of complex networks reviewed in the previous section, these networks have been also applied in other fields of Economics. This formalism is particularly useful for describing the introduction of innovations in markets and/or regions, and has been thoroughly used for that purpose. This methodology has also been used for analyzing business networks. Here some contributions devoted to the study of profit optimization under technological renewal are

reviewed, along with dynamic models of oligopoly with R&D externalities on networks, topics on upstream/downstream R&D networks and welfare and the spreading of products in markets or co-workers networks.

Diffusion in complex social networks has been considered by several authors. D. López-Pintado [124] analyzed the spreading of a given behavior in a population by considering mutual neighbor influence in a network of interacting agents by means of a simple diffusion rule. At a mean-field level, she obtained a threshold for the spreading rate for propagation and persistence in populations. This threshold depends on the connectivity distribution of the underlying social network as well as on the selected diffusion rule. More recently [125], the same author considered the spread of free-riding behavior in social networks introducing a model for a social network with free-riding incentives, where agents are allowed to decide whether or not to contribute to the provision of a given local public good. By means of equilibria analysis of the induced game, the author reported the influence of the degree distribution of the underlying network in the fraction of free-riders. Moreover, López-Pintado and Watts [126] addressed the problem of the collective behavior of individuals facing a binary decision under the influence of their social network. The authors reported both the equilibrium and non-equilibrium properties of the collective dynamics and a response function under global and anonymous interactions.

Concerning business structure the work of Semitiel-García and Noguera Menéndez [127] is relevant, who, using network theory and social network analysis analyzed the influence of inter-industrial structures and the location of economic sectors, on the diffusion of knowledge and innovation. Specifically, they studied the structure and dynamics of the Spanish Inputoutput system over a thirty-five-year period.

Business networks have also been considered by Souma et al. [128], who categorized them into bipartite networks, showing the possibility that business networks will fall into the scale-free category. By means of a one-mode reduction the authors were able to approximately calculate the clustering coefficient and the averaged path length for bipartite networks. These quantities were calculated for networks of banks and companies before/after a bank merger, and they reported quantitative evidence that banks merging increases the cliquishness of companies, and decreases the path length between two companies.

In [129–131] Cerqueti and Rotundo developed models for a set of firms producing a single commodity dealing with the optimal time for the renewal of the technology. Such models consider the aggregate outcome. Eventually, the presence of a hierarchical network organization among firms allows the leader company to propose a financial strategy, but the proposal is followed by the firms at the peripheral of the network with a certain probability only. Depending on the connection level among the companies conditions are obtained for the best strategies to optimize the profit of a district when a technological renewal takes place. The papers refer to empirical results drawn on the most large databases CENSUS and COMPUSTAT for shaping the density of companies and the studies are well suitable for the development of policies for industrial districts [132–134]

With respect to the use of complex networks in oligopoly analysis, in a series of papers, Bischi and Lamantia [135] considered the possibility of reducing the cost of knowledge gain for firms through sharing R&D as well as through investments in R&D cost-reducing activities. These researchers introduced a two-stage oligopoly game for which they analyzed the existence and stability of equilibria in a given network divided into sub-networks. In pursuing such considerations, the authors considered, in the framework of the two-stage oligopoly game, the influence of the degree of collaboration and spillovers on profits, social welfare and overall efficiency [135]. Analytical results are provided for two relevant cases performing numerical experiments and emphasizing the role of the level of connectivity (i.e. the collaboration attitude) inside networks. The effects of unintentional knowledge spillovers inside each network and between competing networks are also considered in [136].

The endogenous formation of *upstream* R&D networks have been studied in a vertically related industry and the welfare implications thereof by Kesavayuth et al. [137]. The authors reported that in the situation where *upstream* firms fix prices, the complete network of firms reaches an equilibrium. In contrast, if upstream firms set quantities, a complete network arise only for sufficiently low R&D spillovers between the firms. If these R&D spillovers are sufficiently high, a partial network arises. Hence, socially optimal equilibrium networks are only reached if upstream firms set prices, and the actual behavior of upstream firms must be taken into account when designing technology policy, and not only the size intra-network R&D spillovers [137]. The downstream firms incentives in a vertically organized industry have also been examined in [138], where the authors analyzed how and when to invest in cost-reducing R&D, and to form a Research Joint Venture (RJV). The authors identified conditions for an RJV to be beneficial to society and discussed integrated innovation and competition policies.

Dal Forno and Merlone [139] have considered a network of individuals supposing that they could propose and successfully implement their best project. Important elements in the network are: (i) mutual knowledge, (ii) agent coordination in choosing the project to implement, (iii) the number of leaders, and (iv) their location. Leaders increase the social network of other agents making possible projects otherwise impossible; at the same time, they are crucial in setting the pace of a balanced expansion of the social matrix. According to evidence, leaders are not those with the greatest number of connections. The presence of leaders provides a solution to the selection problem when there are multiple equilibria.

Finally, Pombo et al. [140] presented evidence of the existence of imitative behaviour among family practitioners in Galicia (Spain), and they used complex network theory and the Ising model (see Sect. 2) in order to describe the entry of new drugs in the market, treating doctors as spins (nodes) in a Watts-Strogatz network. Related to this, one could mention research work done on self-citations of coauthors as defining their research field flexibility, curiosity and in some sense creativity [141–143]. A combination of such investigations might not only lead to new methods for detecting scientists field mobility, but also indicate pertinent features on new ideas related to the evolution of the production of new goods.

1.5 Regional trade, mobility and development

Regional trade is another field of economics that has been extensively treated within the framework of complex network formalism. Specifically, a lot of effort has been devoted to the description of the structure [76, 79, 80, 82, 144, 145], communities [146] and dynamics [81] of the global trade network. Specifically, in the recent past Fronczak and Fronczak [144] reported a statistical mechanics study of the international trade network showing that this network is a maximally random weighted network, and that the product of the GDP's of the trading countries is the only characterizing factor of the directed connections associated to bilateral trade volumes. Moreover, Reyes et al. [147] considered the bilateral trade data from the networks perspective, concluding that there seems to be a cyclical pattern in the regional trade agreement formation on the community structure of the world trade network. From this perspective, the pattern of international integration followed by East Asian countries and its comparison with the Latin American performance has been also reported recently [148].

In this subsection we report contributions that, although not truly devoted to network analysis, they depend on the existence of a (fully connected) network.

Among the empirical and quantitative studies, Paas and Schlitte [149] studied the regional income inequality and convergence process in the EU-25. Paas and Vahi [150] considered the contribution of innovation to regional disparities and convergence in Europe using empirical GDPpc and innovation indicators of the EU-27 NUTS2 the regions. Using principal components factor analysis, three composite indicators of regional innovation capacity were extracted, showing that ca. 60% of variability of regional GDP per capita is associated to regional innovation performance. Regional innovations are seen to promote the increase of inter-regional differences in the short-run. Consequently, further policy interventions beyond innovation activities should be effectively implemented. In this respect see also the related work by Gligor and Ausloos, already mentioned [89–91], on globalization and hierarchical structures in EU, as well as the need of considering appropriate forward and backward correlations within appropriate time intervals [86, 88].

It is common understanding that international mobility of people and workers are increasing globally. According to Paas and Halapuu [151], an ethnically and culturally diverse population is expected to create greater variability in the demand for goods and services as well as in the supply of labour through different skills and business cultures favoring new business activities and future economic growth. The authors state that “although not all immigrants are well-educated and highly-skilled to provide a sufficiently innovative and creative labour force, national economic policies should create conditions that support the integration of ethnic diversity in order to create stable and peaceful environment for economic and political development”. Paas and Halapuu aim at clarifying the possible determinants of peoples’ attitudes towards immigrants depending on their personal characteristics, as well as attitudes towards households socio-economic stability and a country’s institutions. The study’s overwhelming aim is to provide empirical evidence-based reasons for policy proposals that, through integration of ethnically diverse societies, creates a

favorable climate supporting economic growth. Based on the formulated aims, Paas and Halapuu [151] used principal component factor analysis and micro-econometric methods data from the European Social Survey (ESS) fourth round database to examine the attitudes of European people towards immigrants. These attitudes of the European people's towards immigration -which strongly constraints mobility between regions- , these are proved to vary depending on several factors such as the personal characteristics of the respondents, the attitudes towards the country's institutions and socio-economic security, and, finally, country specific conditions.

[152].

On the analytic side, the work of Vitanov and Ausloos [152] is noteworthy. Even though the authors are not considering a network *per se*, they nevertheless include spatial gradients between regions in order to study issues such as: knowledge epidemics that take into account population dynamics and models describing the diffusion of ideas. This work relies on the use of the Lotka-Volterra system of equations with spatial gradients between regions with the addition of demographic input.

1.6 Other social network models

As outlined here above, network theory is increasingly gaining acceptance in the economics community in order to understand the Economy as an evolving complex structure of widely interacting heterogeneous agents [14]. The alternative view to that is to consider the economy as an archipelago of *homo economicus* individuals, still interacting, but on a "shorter range", underlying the neoclassical economy [153] mainstream features. However, in Sociology and other disciplines like Ecology, Computation, etc. this has been so for decades.

In fact, the origin of considering a society as made of interacting entities goes back to Comte [154, 155], the founder of the discipline of sociology, having introduced the term as a neologism¹ in 1838, among other scientific contributions. Note that Comte had earlier used the term "social physics", but that term had been appropriated by others, e.g. Quetelet [157, 158]. Thereafter, Boltzmann and Maxwell imagined similar concepts for describing matter and natural phenomena. Needless to say that much work has followed since.

Many examples of applications of the network approach to social sciences can be found in the literature. Reviews of this work can be found in Kirman [179], Stauffer [159], in a compendium [160] and in the recent book [161] further elaborating the work in Galam [162]. Other studies have focused on computational techniques [163]. In more recent times the approach has been widely accepted and acknowledged [164]. The various methods used by physicists, applied mathematicians, economists, social scientists, differ much from each other due to the targets and methods of analysis. For instance, it is the law of large numbers which allows the application of statistical physics methods [163]. As far as the application of net-

¹ The word was first coined by Sieys in 1780 [156].

works for social modelling is concerned, they contribute to develop tools that allow social scientists to understand how and when social factors such as peer influences, role models, or norms affect individual choices [165, 166]. In what follows, several applications of complex networks are reviewed all aiming at describing some type of social networks.

A large number of studies have applied complex networks to the study of systems like the Internet and the World Wide Web (WWW), and they have been extensively summarized in the reviews cited earlier. However, for an updated review focused specifically on applications the reader is referred to [167], where applications of complex networks to real-world problems and data are reported. The authors surveyed the applications of complex networks formalism in "no less than 11 areas, providing a clear indication of the impact of the field of complex networks". Moreover, the book by Vega-Redondo [168] provides a comprehensive coverage with applications of complex networks to labor markets, peer group effects, trust and trade, and research and development.

Social networks are known to be organized into densely connected communities, with a high degree of the clustering, and being highly assortative. The formation of complex networks has been reported from the experimental perspective by Bernasconi et al. [169] using non-cooperative games of network formation based on the Bala and Goyal type [170]. Toivonen et al. [171] reported a realistic model for an undirected growing network for its use in sociodynamic phenomena. On the other hand, Boguñá et al. [172] used an abstraction of the concept of social distance to define a class of models of social network formation. The evolution of structure within large online social networks is examined in [173], with specific attention focused on the Flickr and Yahoo's social network, showing their segmentation, and providing a detailed characterization of the structure and evolution of their different regions as well as a simple model of network growth capable of mimicking this structure [173]. Finally, Palla et al. [174] quantified social group evolution by means of an algorithm based on clique percolation for the time dependence of overlapping communities on a large scale. Finally, it is noteworthy that the problem of the determination of the community structure in the presence of unobserved structures among the nodes -a rather common situation in social and economic networks- has been addressed by Copic et al. [175], who axiomatically introduced a maximum-likelihood-based method of detecting the latent community structures from network data.

Opinion formation in social systems has also been a matter of great concern in network literature. Apart from some pioneering work like that of Kirman and collaborators in the 80's using the diameter-2 Bollobás model [176], the field has evolved only recently when a plethora of contributions have been reported. As a very recent example, Koulouris et al. [177] reported the multi-equilibria regulation of opinion formation dynamics.

In opinion formation models the single vote of an individual can be influenced and can change, but when the final target is the aggregate, the sampling of many randomly selected people can give a reasonable impression for an upcoming election [178]. In economics agent-based models have been used for the analysis of the

aggregate behaviour of a large number of individuals as model with heterogeneous agents are gaining a more prominent role relative to those with a representative agent [179].

Network theory has been also applied to other social networks of interest like opinion formation, social entrepreneurship, etc. Dal Forno and Merlone adapted the notion of density of a graph to multiple projects and non-dichotomous networks. An appropriate visualization procedure has been implemented in [180]. Social entrepreneurship effects on the emergence of cooperation in networks have been examined in [181], where differences between social entrepreneurs and leaders are analyzed and where the network of interactions may allow for the emergence of cooperative projects. The model reported by the authors consists on two coupled networks standing for knowledge and cooperation among individuals respectively. Any member of the community can be a social entrepreneur. On the basis of this theoretical framework, the authors prove that a moderate level of social entrepreneurship is enough for providing a certain coordination on larger projects, suggesting that a moderate level of social entrepreneurship would be sufficient.

Lambiotte and Ausloos in [182] analyzed the coexistence of opposite opinions in a network with communities. Applying the majority rule to a topology with two coupled random networks, they reproduced the modular structure observed in social networks. The authors analytically calculated the asymptotic behavior of the model deriving a phase diagram that depends on the frequency of random opinion flips and on the inter-connectivity between the two communities. Three regimes were shown to take place: a disordered regime, where no collective phenomena takes place; a symmetric regime, where the nodes in both communities reach the same average opinion; and an asymmetric regime, where the nodes in each community reach an opposite average opinion, registering discontinuous transitions from the asymmetric regime to the symmetric regime.

In this same model, Lambiotte et al. [183] have shown that a transition takes place at a value of the interconnectivity parameter, above which only symmetric solutions prevail. Thus, both communities agree with each other and reach consensus. Below this value, the communities can reach opposite opinions resulting in an asymmetric state. They explicitly analyzed the importance of the interface between the subnetworks.

Finally Lambiotte and Ausloos [184] studied collaborative tagging as a tripartite network, analyzing online collaborative communities described by *tripartite* networks whose nodes are persons, items and tags. Using projection methods they uncovered several structures of the networks, from communities of users to genre families.

Finally, two economico-sociological studies are outlined. One pertains to the interaction of small world networks of biased communities, like the neocreationists vs. the evolution defenders. For this analysis, the networks are considered to be directed but with unweighted links [185] and [186]. The other study mentioned above pertains to bipartite networks made of music listeners downloading some music work from the web [52, 53].

1.7 Suggestions for future research

The application of the theory of complex networks, alone or in combination with other theoretical developments of statistical mechanics, can lead to very interesting results in several areas of economics in particular, and of sociophysics in general. Specifically, the elaboration of a model for the fluxes of entrepreneurs, trade and workers between the European regions undoubtedly demands the usage of complex networks together with non linear model for the dynamics of the agents. This treatment should generalize early regional science contributions, which are typically based on flow equations theories of directed diffusion (see for e.g. those in Refs. [46, 187–189] where goods and migration fluxes are governed by conventional systems of diffusion-like partial differential equations). Moreover, this treatment could be very well complemented by a nonlinear model of production and consumption cycles, in the line of Meadows Dynamics of Commodity Production Cycles [190]. This could be done in analogy to what has been done for biological oscillators (see for example Goodwin’s model of enzyme production in Refs. [191, 192] for a classical review on these kind of models). The introduction of spatial inhomogeneities in these models could also provide a new research path for economic geography.

A list of other potentially fruitful research avenues is provided below.

Innovation and renewal of technology. It could be useful to apply complex networks and agent-based models to the analysis of the spreading of technological renewal, R&D incentives and growth, fiscal (regional) rules [193] usefulness of data analyses through rescale range analysis methods, principal component analysis. Moreover, this framework can also be applied to transportation, migration, growing and diversifying nodes of networks; merging and controls of agents; or tendency toward monopoly through lobbying.

Regional trade and development. The network approach can also be applied for introducing relationships like trade barriers, community detection, clusters, hierarchies; policy implications concerning the economic (regional) clusters arising in the presence of Marshallian and other externalities, etc.

Development of database and data mining. *“I would not have thought that the spread (IT/DE) was going to rise again” [IT politician, summer 2012].* Economic and financial theories need to be tested on real world markets. The complexity and large amount of data makes impossible autonomous data collection at the individual level. Moreover, data providers as the Bureau van Dijk or Bloomberg do implement only certain type of data retrieval, and the work that researchers have to do autonomously delays the production of results, and the detection of information that can be used as input for more complex models. Moreover, the increasing costs of subscriptions to data providers, in conjunction with the progressive decrease of national funding, suggest that the development of synergies with data providers is timely.

Statistical mechanics approaches In future works, phase transitions, coupling between magnetic and cristallographic transitions, thermodynamic (through the notion of cost function) vs. geometric (percolation) could be analyzed, including insta-

bilities in necessarily non equilibrium structures (log-periodic oscillations). Moreover, going beyond Ising model (Blume-Emery-Griffiths, and the forgotten ferroelectric models) should be considered, as well as community detection, forward and backward correlations in networks with weighted and directed links (danger of difficulty in interpreting complex eigenvalues of adjacency matrix), network structure construction and evolution, etc.

Economic and financial networks and risk. “*When Belgium sneezes, the world catches a cold*” [<http://phys.org/news/2010-11-belgium-world-cold.html>] Globalization of economic and financial markets, corporate ownership networks, international trade networks, as well as phenomena like tunnelling, cross-ownership, and boards interlock, change dramatically the profile of financial and economic risks pointing out the relevance of the network structure and are topics to be considered in the future. Understanding such phenomena both at the micro and macro level may help the development of policies also at the local level with potential benefits for regional trade and development.

1.8 Conclusions

The range of applications of complex networks formalism is expanding at a fabulous rate, and has been adopted almost in every field of knowledge having to deal with heterogeneous interacting agents and their emergent phenomena. This is the case, particularly, of economics, and even more specifically of economic geography. The present report gathers contributions from different fields and approaches under the common theme of complex networks analysis in socioeconomic models. The statistical mechanics of complex networks have been reviewed together with some computational aspects related to their description. Models specifically developed for examining topics in various areas of economics and finance, such as, for example, regional trade and development have been the object of specific attention, together with contributions devoted to the application of complex networks analysis to social networks in the broad sense.

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